

# Sets of unit vectors with small pairwise sums

Konrad J. Swanepoel \*

Department of Mathematics and Applied Mathematics

University of Pretoria

Pretoria 0002

South Africa

e-mail: konrad@math.up.ac.za

## Abstract

We study the sizes of  $\delta$ -additive sets of unit vectors in a  $d$ -dimensional normed space: the sum of any two vectors has norm at most  $\delta$ . One-additive sets originate in finding upper bounds of vertex degrees of Steiner Minimum Trees in finite dimensional smooth normed spaces (Z. Füredi, J. C. Lagarias, F. Morgan, 1991). We show that the maximum size of a  $\delta$ -additive set over all normed spaces of dimension  $d$  grows exponentially in  $d$  for fixed  $\delta > 2/3$ , stays bounded for  $\delta < 2/3$ , and grows linearly at the threshold  $\delta = 2/3$ . Furthermore, the maximum size of a  $2/3$ -additive set in  $d$ -dimensional normed space has the sharp upper bound of  $d$ , with the single exception of spaces isometric to three-dimensional  $\ell^1$  space, where there exists a  $2/3$ -additive set of four unit vectors.

1991 Mathematics Subject Classification: Primary 46B20. Secondary 52A21, 52B10.

## 1 Introduction

Let  $X$  be a real normed space of dimension  $d \geq 1$ , with norm  $\|\cdot\|$ . Let  $0 < \delta < 2$ . A  $\delta$ -additive set in  $X$  is a set of unit vectors  $S$  satisfying  $\|x + y\| \leq \delta$  for distinct  $x, y \in S$ . We define  $N_X(\delta)$  to be the largest cardinality of a  $\delta$ -additive set in  $X$ . These notions originate in the analysis of geometric Steiner Minimum Trees in combinatorial optimization. If  $X$  is smooth, then  $N_{X^*}(1)$  is an upper bound for the maximum degree in any Steiner Minimal Tree in  $X$ , where  $X^*$  is the dual of  $X$ ; see [3, 4]. It is easily seen that  $N_X(\delta)$  is finite for all  $\delta \in (0, 2)$ . In this note we investigate the maximum of  $N_X(\delta)$  over all  $X$  of a fixed dimension, keeping  $\delta$  fixed. Let  $N_d(\delta) = \max N_X(\delta)$ , where the maximum is over all  $X$  of dimension  $d$ .

First of all, note that if  $0 < \delta < 2/3$ , then the triangle inequality immediately gives  $N_X(\delta) = 2$  for all  $X$ . If  $\delta > 2/3$ , the following generalization of [2, Theorem 2.4] shows that  $N_d(\delta)$  grows at least exponentially in  $d$ .

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\*I thank the referee for remarks leading to an improved paper.

**Theorem 1** For all  $\delta > \frac{2}{3}$  there exist  $\epsilon = \epsilon(\delta) > 0$  such that for all sufficiently large  $d$  there exists a  $d$ -dimensional normed space  $X$  such that  $N_X(\delta) > (1+\epsilon)^d$ .

By packing arguments it is easily seen that there is also an upper bound for  $N_d(\delta)$  exponential in  $d$  for fixed  $\delta$ . The following is the best such an upper bound we have.

**Theorem 2** If  $X$  is a  $d$ -dimensional normed space, then

$$N_X(\delta) \leq 2 \left( \frac{2}{2-\delta} \right)^d.$$

The proof (in Section 2) uses the Brunn-Minkowski inequality.

Surprisingly, in the remaining case of  $\delta = 2/3$ ,  $N_d(\delta)$  grows linearly in  $d$ . The following theorem gives the exact value of  $N_d(\delta)$ , and shows that for  $d = 3$ , spaces isometric too three-dimensional  $\ell_1$  are exceptional. This theorem can therefore be considered to be a characterization of the three-dimensional affine regular octahedron in the collection of all centrally symmetric convex bodies of any finite dimension.

**Theorem 3** Let  $X$  be a  $d$ -dimensional normed space.

If  $d \neq 1, 3$ , then  $N_X(2/3) \leq d$ , equality being attained e.g. if the unit ball of  $X$  is a cube.

If  $d = 3$ , then  $N_X(2/3) \leq 4$ , with equality iff the unit ball of  $X$  is an affine regular octahedron.

In the sequel we fix our  $d$ -dimensional space to be  $\mathbb{R}^d$  with inner product  $\langle \cdot, \cdot \rangle$ . For  $S \subseteq \mathbb{R}^d$  let  $\text{conv } S$  be the convex hull of  $S$ . If  $S$  is Lebesgue measurable, let  $\text{vol}(S)$  be the Lebesgue measure or *volume* of  $S$ . Let  $\ell_\infty^d$  be  $\mathbb{R}^d$  with norm  $\|(x_1, \dots, x_d)\|_\infty = \max_i |x_i|$ , and  $\ell_1^d$  be  $\mathbb{R}^d$  with norm  $\|(x_1, \dots, x_d)\|_1 = |x_1| + \dots + |x_d|$ . Note that the unit ball of  $\ell_\infty^d$  is a  $d$ -dimensional cube. Also note that a three-dimensional normed space is isometric to  $\ell_3^1$  iff its unit ball is an *affine regular octahedron* centered at 0, i.e. a non-singular linear image of the regular octahedron centered at 0.

We now state a technical lemma important to the proofs of Theorems 1 and 3 (sections 4 and 3, respectively). The lemma reduces the existence of norms admitting a given  $\delta$ -additive set to a set of linear inequalities. The proof is in Section 5.

**Lemma 4** Let  $x_1, \dots, x_m \in \mathbb{R}^d \setminus \{0\}$  and  $0 < \delta \leq 2$ . There exists a norm  $\|\cdot\|$  on  $\mathbb{R}^d$  such that

$$\begin{aligned} \|x_i\| &= 1 \quad \forall i = 1, \dots, m, \\ \|x_i + x_j\| &\leq \delta \quad \forall 1 \leq i < j \leq m, \end{aligned} \tag{1}$$

iff there exist  $y_1, \dots, y_m \in \mathbb{R}^d$  such that

$$\begin{aligned} \langle x_i, y_i \rangle &= 1 \quad \forall i = 1, \dots, m, \\ -1 \leq \langle x_j, y_i \rangle &\leq \delta - 1 \quad \forall \text{ distinct } i, j = 1, \dots, m, \\ -\delta \leq \langle x_j + x_k, y_i \rangle &\leq \delta - 1 \quad \forall i, j, k = 1, \dots, m \text{ with } j \neq k. \end{aligned} \tag{2}$$

## 2 Proof of Theorem 2

Let  $S$  be a  $\delta$ -additive set containing  $N$  vectors. Then for distinct  $x, y \in S$ ,  $\|x + y\| \leq \delta$ ,  $\|x - y\| \geq 2 - \delta$ . We partition  $S$  into two sets  $S_1$  and  $S_2$  of sizes  $\lfloor N/2 \rfloor$  and  $\lceil N/2 \rceil$ , respectively. Let  $V_i := \bigcup_{x \in S_i} B(x, 1 - \delta/2)$  for  $i = 1, 2$ . Each  $V_i$  consists of closed balls with disjoint interiors, and therefore,  $\text{vol}(V_1) = (\lfloor N/2 \rfloor)2^{-d}\text{vol}(B)$  and  $\text{vol}(V_2) = (\lceil N/2 \rceil)2^{-d}\text{vol}(B)$ . Also,  $V_1 + V_2 \subseteq B(0, 3 - \delta)$ . By the Brunn-Minkowski inequality (see [1]) we obtain

$$(\lfloor N/2 \rfloor^{1/d} + \lceil N/2 \rceil^{1/d})(1 - \delta/2) \leq 3 - \delta,$$

and the result follows.  $\blacksquare$

## 3 Proof of Theorem 3

To see that equality may hold for  $d \geq 4$ , consider the space  $X = \ell_\infty^d$  with unit ball  $[-1, 1]^d$ , and let  $S$  be the set of all coordinate permutations of

$$(1, -\frac{1}{3}, -\frac{1}{3}, \dots, -\frac{1}{3}) \in \ell_\infty^d.$$

Then  $S$  is a set of  $d$  unit vectors and  $\|x + y\| = \frac{2}{3}$  for distinct  $x, y \in S$ .

For  $d = 3$ , note that the 4 vectors of  $S$  in  $\ell_\infty^4$  are all in the hyperplane  $\{x \in \mathbb{R}^4 : \sum_i x_i = 0\}$ , and thus in a 3-dimensional subspace. It is easy to see that this subspace is isometric to  $\ell_1^3$ , with unit ball the octahedron

$$\text{conv}\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

We first derive the upper bound for  $d \neq 1, 3$ . Let  $\{x_1, \dots, x_m\}$  be a  $\frac{2}{3}$ -additive set of unit vectors in  $X$ . We suppose for the sake of contradiction that  $m = d + 1$ . By Lemma 4 there are  $y_1, \dots, y_m \in X$  such that

$$\langle x_j, y_i \rangle \leq -\frac{1}{3} \leq \frac{1}{2} \langle x_j + x_k, y_i \rangle \quad \forall i, j, k \text{ with } j \neq k, j \neq i.$$

Thus  $\langle x_j, y_i \rangle \leq \langle x_k, y_i \rangle$ . But similarly,  $\langle x_k, y_i \rangle \leq \langle x_j, y_i \rangle \quad \forall i, j, k \text{ with } k \neq j, k \neq i$ . Thus  $\langle x_j, y_i \rangle = \langle x_k, y_i \rangle = -\frac{1}{3} \quad \forall \text{ distinct } i, j, k$ , i.e.

$$\langle x_j, y_i \rangle = \begin{cases} 1 & \text{if } i = j, \\ -\frac{1}{3} & \text{if } i \neq j. \end{cases} \quad (3)$$

Note that we have used the fact  $m \geq 3$ .

Suppose that  $\lambda_i$  is a sequence of scalars for which  $\sum_{i=1}^m \lambda_i x_i = 0$ . Then for all  $j$  we have

$$0 = \sum_{i=1}^m \lambda_i \langle x_i, y_j \rangle = \frac{4}{3} \lambda_j - \frac{1}{3} \sum_{i=1}^m \lambda_i.$$

Thus all  $\lambda_j$ 's are equal:

$$\lambda_j =: \lambda \quad \forall j = 1, \dots, m, \quad (4)$$

and  $(\frac{4}{3} - \frac{1}{3}m)\lambda = 0$ . Thus  $x_1, \dots, x_m$  are linearly independent, since  $m \neq 4$ , contradicting  $m = d + 1$ .

We now treat the remaining case  $d = 3$  (as  $d = 1$  is trivial). By considering  $X$  as a subspace of some 4-dimensional space, we immediately obtain from the previous argument that  $N_X(2/3) \leq 4$ . Alternatively, we can argue directly as follows. Using the John-Loewner ellipsoid (see e.g. [5, Theorem 3.3.6]), we obtain an inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|_2 = \sqrt{\langle \cdot, \cdot \rangle}$  in  $X$  such that

$$\|x\| \leq \|x\|_2 \leq \sqrt{3}\|x\|.$$

Let  $S = \{x_1, \dots, x_m\}$  be a  $\frac{2}{3}$ -additive set of unit vectors in  $X$ . Then  $\|x\|_2 \geq 1$  for all  $x \in S$ , and  $\|x + y\|_2^2 \leq 3\|x + y\|^2 \leq \frac{4}{3}$  for all distinct  $x, y \in S$ . Thus  $\frac{4}{3} \geq 2 + 2\langle x, y \rangle$ , and  $\langle x, y \rangle \leq -\frac{1}{3}$ . It follows that  $\|\sum_{i=1}^m x_i\|_2^2 \leq m - \frac{2}{3}\binom{m}{2}$ , implying  $m \leq 4$ .

It remains to show that  $\ell_1^3$  is the unique 3-dimensional Minkowski space admitting four unit vectors satisfying  $\|x_i + x_j\| \leq \frac{2}{3}$  for  $i \neq j$ . Suppose we have four such vectors for some norm  $\|\cdot\|$ . From the triangle inequality it follows that  $\|x_i + x_j\| = \frac{2}{3}$  for  $i \neq j$ , i.e.  $z_i := \frac{3}{2}(x_i + x_j)$ ,  $i = 1, 2, 3$  are unit vectors that are furthermore linearly independent: If  $\sum_i \lambda_i z_i = 0$  then  $\sum_i \lambda_i \langle z_i, y_j \rangle = 0$  for each  $j$ , and from (3) it follows after some calculation that all  $\lambda_i = 0$ .

Since  $x_1, \dots, x_4$  are linearly dependent, we have  $\sum_i \lambda_i x_i = 0$  for some  $\lambda_i$  not all 0. From (4) it follows that the  $\lambda_i$  are equal, and therefore,  $\sum_i x_i = 0$ . Hence  $z_1 + z_2 + z_3 = \frac{3}{2}(x_1 + x_2 + x_3 + 3x_4) = 3x_4$ , and  $\|\frac{1}{3}(z_1 + z_2 + z_3)\| = 1$ . Thus,  $\text{conv}\{z_1, z_2, z_3\}$  is a face of the unit ball.

Similarly,  $\|\frac{1}{3}(\epsilon_1 z_1 + \epsilon_2 z_2 + \epsilon_3 z_3)\| = 1$  for any  $\epsilon_i = \pm 1$ , and

$$\text{conv}\{\epsilon_1 z_1, \epsilon_2 z_2, \epsilon_3 z_3\}, \epsilon_i = \pm 1$$

are all faces of the unit ball, which must therefore be the affine regular octahedron  $\text{conv}\{\pm z_1, \pm z_2, \pm z_3\}$ . It follows that  $\|\sum_i \lambda_i z_i\| = \|(\lambda_1, \lambda_2, \lambda_3)\|_1$ .

Note that it is easy to calculate

$$\begin{aligned} x_1 &= \frac{1}{3}(z_1 - z_2 - z_3), & x_2 &= \frac{1}{3}(-z_1 + z_2 - z_3), \\ x_3 &= \frac{1}{3}(-z_1 - z_2 + z_3), & x_4 &= \frac{1}{3}(z_1 + z_2 + z_3). \end{aligned}$$

Except for reflections in the coordinate planes or permutations of the coordinates (i.e. linear isometries of  $\ell_1^3$ ), these vectors form the unique  $2/3$ -additive set of four points in  $\ell_1^3$ . They are the centroids of four alternate faces of the octahedron, and the vertex set of a regular tetrahedron. ■

## 4 Proof of Theorem 1

Let  $\delta' := (3\delta - 2)/(6 - \delta) > 0$ . (Recall that we assume  $\delta < 2$ .) By a result of [6] it is possible to find at least  $(1 + \epsilon)^d$  euclidean unit vectors  $v_i$  such that  $|\langle v_i, v_j \rangle| < \delta'$  for all distinct  $i, j$ . Regard  $\mathbb{R}^{d+1}$  as  $\mathbb{R}^d \oplus \mathbb{R}$ , identify  $\mathbb{R}^d$  with the

$\mathbb{R}^d$  factor, and let  $e$  be a unit vector orthogonal to  $\mathbb{R}^d$ . Let  $x_i := v_i + e$  for all  $i$ . We now check that Lemma 4 may be applied with  $y_i := \lambda v_i + (1 - \lambda)e$ , where  $\lambda = \frac{2}{3} - \frac{\delta}{4}$ .

Firstly, for all  $i$ ,

$$\langle x_i, y_i \rangle = \lambda + 1 - \lambda = 1.$$

Secondly, for all  $i \neq j$ ,

$$\langle x_j, y_i \rangle = \lambda \langle v_i, v_j \rangle + 1 - \lambda \begin{cases} \leq \lambda \delta' + 1 - \lambda = \delta - 1, \\ \geq -\lambda \delta' + 1 - \lambda = -\delta/2 \geq -1. \end{cases}$$

Thus for all distinct  $i, j$ ,

$$\langle x_i + x_j, y_i \rangle = 1 + \langle x_j, y_i \rangle \geq 1 - 1 \geq 0 \geq -\delta.$$

Thirdly, for all  $i, j, k$  such that  $j \neq i, k \neq i$ ,

$$\langle x_i + x_k, y_i \rangle \geq -\delta/2 - \delta/2 = -\delta.$$

Lemma 4 now gives the required norm. ■

## 5 Proof of Lemma 4

$\Rightarrow$  We choose norming functionals  $y_i$  for  $x_i$ :

$$\langle x_i, y_i \rangle = 1 = \sup_{\|x\|=1} \langle x, y_i \rangle.$$

Then for  $i \neq j$  we have

$$1 + \langle x_j, y_i \rangle = \langle x_i + x_j, y_i \rangle \leq \|x_i + x_j\| \leq \delta,$$

and hence  $\langle x_j, y_i \rangle \leq \delta - 1$ . Also,  $-\langle x_j, y_i \rangle \leq \|-x_j\| = 1$ , and therefore,  $\langle x_j, y_i \rangle \geq -1$ . Also, for  $j \neq k$  and  $i$ ,

$$\langle -x_j - x_k, y_i \rangle \leq \|-x_j - x_k\| \leq \delta,$$

and  $\langle x_j + x_k, y_i \rangle \geq -\delta$ .

$\Leftarrow$  Let  $K$  be the convex hull of

$$\{\pm x_i : i = 1, \dots, m\} \cup \{\pm \frac{1}{\delta}(x_i + x_j) : 1 \leq i < j \leq m\}.$$

Then  $K$  is centrally symmetric and convex. If  $K$  has no interior points (i.e. if  $K$  is contained in some hyperplane), we “thicken”  $K$ : Let  $V$  be the linear span of  $K$ ,  $U = V^\perp$  and  $e_1, \dots, e_k$  an orthonormal basis for  $U$ . Replace  $K$  by  $K + [-1, 1]e_1 + \dots + [-1, 1]e_k$ . We now have that  $K$  is a centrally symmetric convex body defining a norm

$$\|x\|_K := \inf\{t \geq 0 : x \in tK\}.$$

Obviously,  $\|x_i\|_K \leq 1 \ \forall i$  and  $\|x_i + x_j\|_K \leq \delta \ \forall i \neq j$ . It remains to show that  $\|x_i\| = 1$ , i.e. that each  $x_i$  is on the boundary of  $K$ . It is sufficient to show that the closed half space  $\{x \in \mathbb{R}^d : \langle x, y_i \rangle \leq 1\}$  contains  $K$ . By the definition of  $K$  it is sufficient to show that

$$\pm \langle x_j, y_i \rangle \leq 1 \ \forall i, j \quad (5)$$

$$\text{and } \pm \langle x_j + x_k, y_i \rangle \leq \delta \ \forall i, j, k \text{ with } j \neq k. \quad (6)$$

Now (5) holds, since  $\langle x_i, y_i \rangle = 1$ , and for  $j \neq i$ ,  $-1 \leq \langle x_j, y_i \rangle \leq \delta - 1 \leq 1$ . Also,

$$\langle x_i + x_j, y_i \rangle = 1 + \langle x_j, y_i \rangle = 1 + \langle x_j, y_i \rangle \begin{cases} \leq 1 + \delta - 1 = \delta \\ \geq 1 - 1 > -\delta \end{cases}$$

for  $i \neq j$ , and

$$\langle x_j + x_k, y_i \rangle \begin{cases} \leq \delta - 1 + \delta - 1 \leq \delta \\ \geq -\delta \end{cases}$$

for  $j \neq i, k \neq i$ , and so (6) holds. ■

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